Dynamics of Meteorological Fronts

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front begins to overtake the warm front, leading ultimately
to occlusion. One may expect that some qualitative features
of the first part of the propagation could be obtained from
a linearized small-perturbation procedure, and this has been
investigated by various authors (see Haurwitz [5], p. 307);
however, the occlusion process and the events leading up to
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Sec. 1. Introduction

In this paper, an alternative treatment is given of the problem described by Professor Stoker in his report "Dynamical Theory for Treating the Motion of Cold and Warm Fronts in the Atmosphere." The description of the phenomena which we wish to explain and the formulation of the basic mathematical equations are discussed in detail in that report and will not be repeated here. The present method starts from the same basic equations but differs in the way in which these equations (which are too difficult to solve without further simplification) are approximated.

The propagation of disturbances on a frontal surface appears to fall into two parts. First, the initial disturbance moves along the discontinuity surface with its amplitude (i.e. the displacement of the surface normal to its undisturbed position) increasing; then, at a more advanced stage, the amplitude remains roughly constant and the cold front begins to overtake the warm front, leading ultimately to occlusion. One may expect that some qualitative features of the first part of the propagation could be obtained from a linearized small-perturbation procedure, and this has been investigated by various authors (see Haurwitz [5], p. 307); however, the occlusion process and the events leading up to it are certainly nonlinear phenomena. This later development is considered here, and an approximate treatment, which seems to be capable of explaining some of the results of observation, is presented.

It should be remarked at the outset that, in view of the complexity of the problem, any theory which includes some of the main effects or gives some qualitative insight

into the problem is considered to be of value, even though
it may be lacking in other respects, and may not apply in
every situation. The present work should be interpreted in
this light.

Sec. 2. Approximation to the equations of motion

The equations of motion for the components u and v of
the velocity in the cold air and the height, h, of the
frontal surface are

(1) $$h_t + (uh)_x + (vh)_y = 0,$$

(2) $$u_t + uu_x + vu_y - 2\omega v \sin \varphi = -kh_x,$$

(3) $$v_t + uv_x + vv_y + 2\omega \sin \varphi (u - \frac{\rho_1 u_1}{\rho_0}) = -kh_y.$$

(ρ_0 and ρ_1 are the densities of the cold and warm air, and
u_0 and u_1 are the undisturbed velocities; ω is the angular
velocity of the earth, φ is the latitude, and $k = g(1 - \frac{\rho_1}{\rho_0})$.)
In the undisturbed state, $u = u_0$, $v = 0$, $h = h_0$; hence, the
well-known Margules formula is obtained:

(4) $$\alpha = \frac{dh_0}{dy} = \frac{2\omega \sin \varphi}{k}(\frac{\rho_1 u_1}{\rho_0} - u_0).$$

In general, u_0 and u_1 may be functions of y but it will of-
ten be convenient to take the simple case of constant u_0 and
u_1 and to neglect the variation of φ with y, so that α is
constant (i.e. the cold air forms a wedge of angle α).

Now, for the situation which actually arises, α is
small (a typical value being 1/200) and the theory developed
here is an attempt to utilize this fact. It is assumed that
in the disturbed motion the rates of change of the flow
quantities in the direction of y remain small and of the
same order as α; on the other hand, h_x and u_x may be finite
compared to α, and we are interested in non-small values of

them for otherwise we would be reduced to something like a
linear theory. Then, in view of the small change of h in
the y-direction and the smallness of the Coriolis term, it
is expected that the main motion will be in the x-direction,
i.e. the "crossflow," v, will be of smaller order than u.
Thus, approximations are made that y-derivatives are of
smaller order than x-derivatives and that v/u is small.
More precisely, suppose $h_y/h_x = 0(\alpha)$; then, since
$2\omega u_0 \sin \varphi = 0(k\alpha)$ from (4), equations (2) and (3) show
that $Dv/Dt \approx \alpha\, Du/Dt$. Hence, unless v builds up over a
large period of time whereas u remains bounded, v/u will be
$0(\alpha)$. Moreover, consistent with $h_y/h_x \approx 0(\alpha)$, we should
expect $u_y/u_x = 0(\alpha)$, $v_y/v_x = 0(\alpha)$. With these approxima-
tions, the ratios $vu_y/(vu_x)$, $vv_y/(uv_x)$, and $(vh)_y/(uh)_x$ are
$0(\alpha^2)$, and, from (4), $2\omega v \sin \varphi/kh_x = 0(\dfrac{v}{u_0}\dfrac{h_y}{h_x}) = 0(\alpha^2)$;

therefore, neglecting these small terms (which are of the
second order in small quantities), the equations of motion
become

(5) $$h_t + uh_x + hu_x = 0,$$

(6) $$u_t + uu_x + kh_x = 0,$$

(7) $$v_t + uv_x = 2\omega \sin \varphi (\frac{\rho_1 u_1}{\rho_0} - u) - kh_y.$$

In the first two equations, y now enters only as a parameter,
but the solution depends strongly on y, as will be seen,
through the conditions $(u \rightarrow u_0$, $h \rightarrow h_0(y)$ as $|x| \rightarrow \infty)$
under which the equations are solved. When u and h are
known, v is found from the linear equation (7).

Of course, in many cases the approximations made above
may not be true; for example, they would not apply when the
amplitude of the disturbance is small compared to its wave
length, and thus could not be used to study the initial

formation of frontal disturbances. Again, it may be that v
does grow continually until it is no longer negligible. How-
ever, the approximate equations (5), (6), (7) do exhibit
some features of the motion which are expected to be present
even when other effects (neglected by them) also play an im-
portant role.

Sec. 3. Solution of the approximate equations

Equations (5) and (6) have already been studied in
great detail, since, on substituting $c^2 = kh$, they are iden-
tical with the equations for the velocity u and sound speed
c in the one-dimensional unsteady flow of a gas with ratio
of specific heats equal to 2. With $k = g\rho$ they are the
equations for waves on shallow water of density ρ (see [6]);
with the present k they have been used by Freeman [3] to
discuss waves on a discontinuity surface when there is no
dependence on y, i.e. the undisturbed surface is at a uni-
form height above the ground and the disturbance is a func-
tion of x and t only. In the problem we are considering, we
have essentially a Freeman type problem to solve in each
plane y = constant, with the initial conditions depending on
the parameter y. Now, we consider the motion in any such
plane.

Introducing $c^2 = kh$, (5) and (6) become

$$(8) \qquad 2c_t + 2uc_x + cu_x = 0,$$

$$(9) \qquad u_t + uu_x + 2cc_x = 0,$$

or,

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm 2c)$$

$$= (u_t + uu_x + 2cc_x) \pm (2c_t + 2uc_x + cu_x)$$

$$= 0;$$

hence, u + 2c is "constant"[(#)] on each <u>positive</u> characteristic curve dx/dt = u + c, and u - 2c is "constant" on each <u>negative</u> characteristic curve dx/dt = u - c.[(##)] This characteristic form of the equations already enables any initial value problem to be solved numerically (by the well-known "method of characteristics"), but many important problems can be solved without numerical work by use of the so-called "simple wave" solutions. The latter are solutions for which either u - 2c or u + 2c is a "constant," not only on each of the corresponding set of characteristics separately, but <u>throughout</u> the disturbed region. For example, suppose that u - 2c is "constant" in the whole region. Then, since in addition u + 2c is a "constant" on each positive characteristic dx/dt = u + c, it follows that u and c must be individually "constant" on each of these characteristics. Hence in the (x,t)-plane these positive characteristics form a set of straight lines, the slopes being determined by the values of u + c on them.

An example of the way in which such simple waves arise may be seen in the following problem (Figure 1, see next page). Suppose that at t = 0, arbitrary values of u and c are given in the range $x_1 \leq x \leq x_2$, and that outside this range u = u_0, c = c_0, where u_0 and c_0 are "constant." For the moment we treat the case in which c is always positive and greater than u. Consider the negative characteristic BCE through x = x_2. It is clear that along it, and along any negative characteristic to the right of it,

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[(#)] Henceforth, quotation marks around the word constant mean that the quantity is independent of x and t; the value of the "constant" depends on y.

[(##)] The case of h = 0, $h_x \neq 0$ at a certain point, will arise later, and it should be noted that these results do not reduce, at c = 0, to u = constant on dx/dt = u, since although c = 0, $c_x = \infty$ and the derivation breaks down; in fact the change of u on dx/dt = u is $u_t + uu_x = -kh_x$.

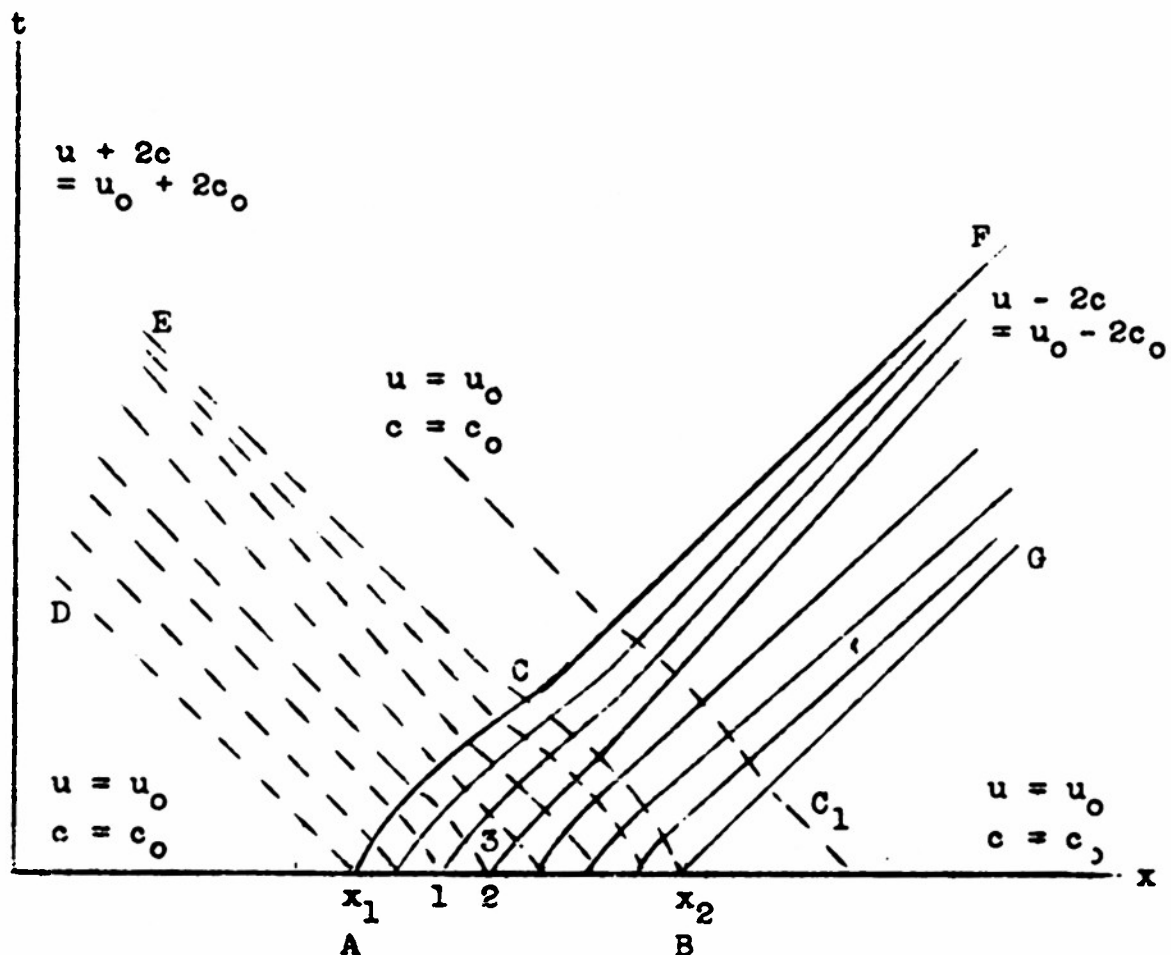$u - 2c = u_o - 2c_o$ since all those negative characteristics

Fig. 1

meet $t = 0$ at points where $u = u_o$, $c = c_o$. Similarly, to
the left of the positive characteristic ACF, $u + 2c = u_o + 2c_o$.
In particular, then, for points in ECF both the above condi-
tions are satisfied, and we have $u = u_o$, $c = c_o$ there. Of
course, the region $x > x_2$ is undisturbed until the first

wavelet, the positive characteristic BG: $x = x_2 + (u_o + c_o)t$, arrives. Hence, the propagation of the disturbance to the right is confined between characteristics BG and CF; $u - 2c = u_o - 2c_o$ in this region so that it is a simple wave with the positive characteristics forming a set of straight lines, and u and c "constant" along each of them. Similarly the propagation to the left is a simple wave between AD and CE, with $u + 2c = u_o + 2c_o$. In the "interaction" region, ABC, it is necessary to use a rather devious numerical method. Two points 1 and 2 (Figure 2) are taken close together on the x-axis where u and c are known, and short
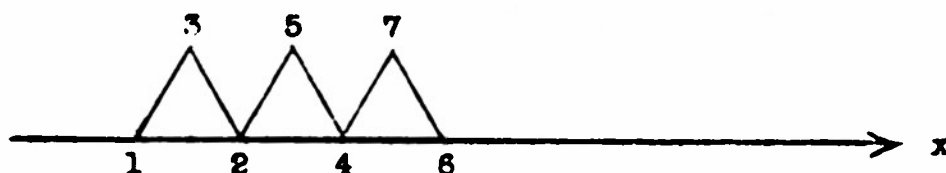

Fig. 2

approximate characteristic segments $dx/dt = u_1 + c_1$, $dx/dt = u_2 - c_2$, are constructed through the points 1 and 2 respectively (here subscripts denote values at the corresponding points); in this way, a point 3, the intersection of the two characteristics, is obtained. The values of u_3 and c_3 satisfy the two equations $u_1 + 2c_1 = u_3 + 2c_3$, $u_2 - 2c_2 = u_3 - 2c_3$ which are the characteristic relations between u and c on the positive and negative characteristics respectively; hence u_3 and c_3 are easily found. The values at 5 can be obtained from 2 and 4, 7 from 4 and 6 and so on.

The solution in the region ABC can then be obtained by repetitions of this process. When the values of u and c are known on BC, the simple wave CFGB is obtained by drawing straight lines through appropriate points on BC with slopes equal to the values of u + c at these points. Similarly the simple wave ADEC may be constructed. When the straight characteristics of the simple wave are converging, the continuous solution is valid only in the region before they overlap. After they overlap a discontinuous jump of u and c across a certain curve must be introduced, analogous to the shocks of gas dynamics and the bores of shallow water theory; this topic is postponed until Section 4.

The above argument showing the separation of an arbitrary initial disturbance into two simple waves applies when c is sufficiently large (i.e. y sufficiently large), but does not apply when the plane y = constant includes points where the frontal surface meets the ground. Consider the case when c = 0 at A and C (Figure 3) and suppose that there
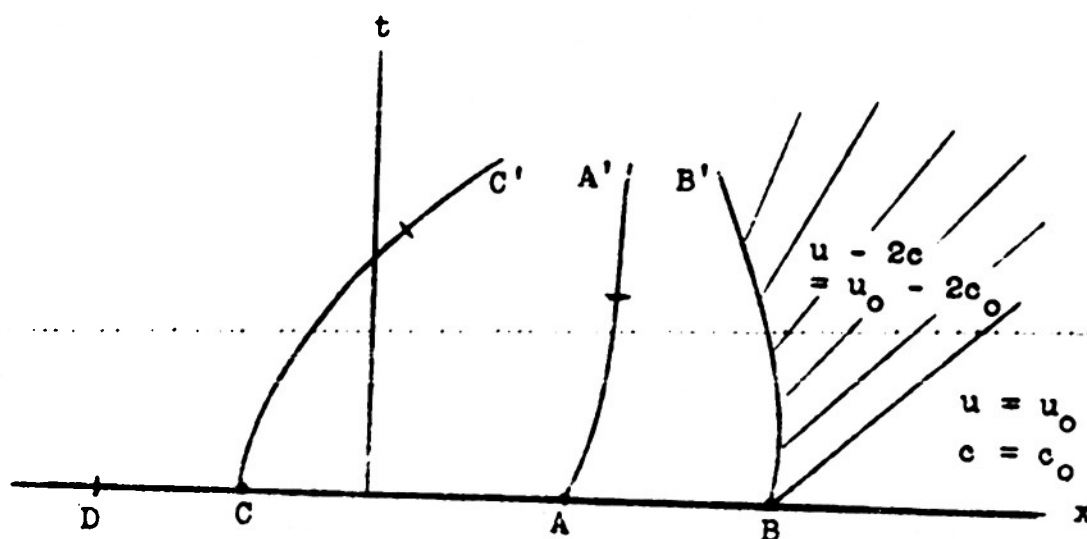

Fig. 3

is no fluid between A and C. The locus in the (x,t)-plane
of the point where $c = 0$ is $dx/dt = u$ since equation (9)
shows that, when $c = 0$, $c_t + uc_x = 0$, i.e. c is constant on
$dx/dt = u$. Thus <u>both</u> characteristic directions $dx/dt = u \pm c$
are tangent to the locus of $c = 0$. Equation (9) shows that
the rate of change of u along the curve is

$$(10) \qquad\qquad u_t + uu_x = -kh_x.$$

Therefore, if $h_x \neq 0$, the characteristic relations $(u \pm 2c =$
"constant") which degenerate into $u =$ "constant" on $c = 0$,
are not satisfied, and the curve $dx/dt = u$ is not a charac-
teristic but is an envelope of both sets of characteristics;
if $h_x = 0$, then u remains constant, and $dx/dt = u$ becomes a
characteristic. Again we can argue that to the right of the
negative characteristic through B, $u - 2c = u_o - 2c_o$, and
there is a simple wave on the positive characteristics; how-
ever, between AA' and BB', the disturbance would have to be
determined numerically.

One general remark may be made here. Along AA', $h_x \geq 0$,
and therefore from (10), u decreases; on CC', $h_x \leq 0$, and
therefore u increases. Of course the values of u will be-
come constant after a certain time, but the result shows at
least some tendency for the gap to close up. This is of
some interest since it can play a role in explaining why oc-
clusion occurs. (Another reason for the overtaking of a
warm front by a cold front will be given in Section 4.)

Now although it is seen that an initial disturbance (in
which u and c are both given arbitrarily) does not separate
completely into two simple waves, the author still feels
that it is relevant to consider the case of a disturbed re-
gion which is entirely a simple wave (for <u>all</u> y, not only
away from the intersection with the ground). After all,
setting up an initial value problem is not necessarily the
correct approach for our problem. The frontal disturbances
are initiated in some complicated way which certainly

involves influences (the interference of a mountain range, for example) which are not included in our basic formulation at all. These may even be more like the piston problems, etc. of gas dynamics which are known to produce simple waves. Moreover, the present theory makes no claim to describe the whole development, and is explicitly directed to the later stages which lead to occlusion. We are dealing here with a fully developed disturbance which may have already separated into the two disturbances propagating in opposite directions (a typical feature of problems of wave propagation). In any case, our object is to see whether the observed phenomena are possible within the framework of our approximate theory; for this purpose it is natural to consider first the simple waves which are known to have great significance in other problems.

In particular the simple wave describing propagation in the direction of increasing x is considered, since fronts are nearly always observed to be moving eastward (in the x-direction) over the United States. The solution of (5) and (6) which is chosen, then, has $u - 2c =$ "constant," i.e.

$$(11) \qquad u - 2c = A(y),$$

say, throughout the region. In addition, as explained before, we know that $u + 2c$ is a function only of y on each positive characteristic $dx/dt = u + c$; hence, u and c are individually functions only of y on each of these characteristics. Therefore, the characteristic equation may be integrated to yield

$$(12) \qquad x = \xi + (u + c)t,$$

where ξ is the value of x at $t = 0$. Now, the values of u and c on the characteristic (12) are exactly the same as the values (for the same y) at the point $t = 0$, $x = \xi$; therefore, if we suppose, for example, that c is a given function

$C(x,y)$ at $t = 0$, the value of c on (12) is $C(\xi,y)$ and the value of u is, from (11), $A(y) + 2C(\xi,y)$. Thus, the complete solution is

$$(13) \quad \begin{cases} c = C(\xi,y), \\ u = A(y) + 2C(\xi,y), \\ x = \xi + \left\{ A(y) + 3C(\xi,y) \right\} t. \end{cases}$$

(Although the arbitrary function occurring in a simple wave may be specified in other ways, it is convenient for our purposes to give the value of h, and hence c, at $t = 0$.) At large distances from the disturbance (i.e. as $|x| \to \infty$), u and c must have their undisturbed values $u_0(y)$ and $c_0(y)$, respectively; hence, $A(y)$ is determined by (11) to be

$$(14) \quad A(y) = u_0(y) - 2c_0(y).$$

It should be stressed that, in using the solution (13), it is much easier to fix attention on individual planes $y = $ constant, study the solutions as functions of x and t in each plane with y as a parameter, and combine them finally to give the full picture.

Turning now to the equation (7) for v, it is convenient (since u and $kh = c^2$ are functions of ξ, y) to change variables from (x,y,t) to (ξ,y,t). Writing the derivatives in the old system on the left and those in the new on the right, we have

$$(15) \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \xi_t \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y} + \xi_y \frac{\partial}{\partial \xi},$$

and, from (13),

$$(16) \quad \xi_t = -\frac{A + 3C}{1 + 3C_1 t}, \quad \xi_x = \frac{1}{1 + 3C_1 t}, \quad \xi_y = -\frac{(A' + 3C_2)t}{1 + 3C_1 t}$$

where $C_1(x,y)$ and $C_2(x,y)$ denote $\partial C(x,y)/\partial x$ and $\partial C(x,y)/\partial y$, respectively. Equation (7) becomes

$$(17) \quad \frac{\partial v}{\partial t} - \frac{C}{1 + 3C_1 t} \frac{\partial v}{\partial \xi} = -\frac{2C}{1 + 3C_1 t}(C_2 - A'tC_1)$$

$$+ 2\omega \sin \varphi \left(\frac{\rho_1 u_1}{\rho_c} - A - 2C\right).$$

The details of the derivation of the solution of this equation are not of sufficient importance to be included; it may easily be verified that the general solution is

$$(18) \quad v = 2\int C_2(\xi,y) \, d\xi + \left\{ C(A' - 6\omega \sin \varphi) \right.$$

$$+ 2\omega \sin \varphi \left(\frac{\rho_1 u_1}{\rho_o} - A\right) \Big\} t$$

$$+ (A' - 2\omega \sin \varphi)\xi + F\left\{ c^3 t + \int c^2 \, d\xi, y \right\},$$

where F is an arbitrary function to be determined from some initial condition. For example, if it is assumed that $v = V(x,y)$ at $t = 0$ then F is specified by

$$(19) \quad F\left\{ \int C^2(x,y) \, dx, y \right\} = V(x,y) - 2\int C_2(x,y) \, dx$$

$$- (A' - 2\omega \sin \varphi)x.$$

The simple wave on the other set of characteristics, with $u + 2c$ a function only of y, and the corresponding value of v may be described similarly.

## Sec. 4. The propagation of warm and cold fronts

The behavior of warm and cold fronts is now discussed on the basis of the simple wave solution established in Section 3. Across the United States, the intersection of the frontal surface with the ground lies roughly east-west, in general; u_1 is positive and u_o may be of either sign (if

$u_o > 0$, it must be less than $\rho_1 u_1 / \rho_o$ in order that $\alpha > 0$ in (4)). A disturbance on the frontal surface appears frequently to be a northward push of warm air and <u>nearly always is observed to propagate eastward</u> (increasing x) so that a warm front is followed by a cold front as shown in Figure 4 (see next page). This being the case, the simple wave solution on the "positive" characteristics would appear to be the relevant one, and is the one which will be described. It may be remarked, however, that as far as the theory developed above is concerned the disturbance could be propagated either on positive or negative characteristics, or both. The propagation velocities $u \pm c$ are positive and negative, respectively, for sufficiently large y (and hence large c), but near the intersection with the ground, where c is small, the direction of propagation would be the same as u in both cases. Hence, if u is usually positive (west wind) there would be a tendency for the fronts observed at the ground to propagate eastward in all cases. There is some evidence that u is more frequently positive than negative. Apart from this small argument, the explanation for the eastward movement of the fronts must lie outside the scope of this theory.

We suppose, now, that there is a simple wave on a set of positive characteristics with $u - 2c = u_o - 2c_o$ throughout, as given by (11) and (14). Immediately an important result is obtained: At the intersection of the surface with the ground where $c = 0$, we have $u = u_f = u_o - 2c_o$. If the approximate values $u_o = $ constant, $\alpha = h_o'(y) = $ constant, $c_o = \sqrt{kh} = \sqrt{\alpha k y}$, are taken,

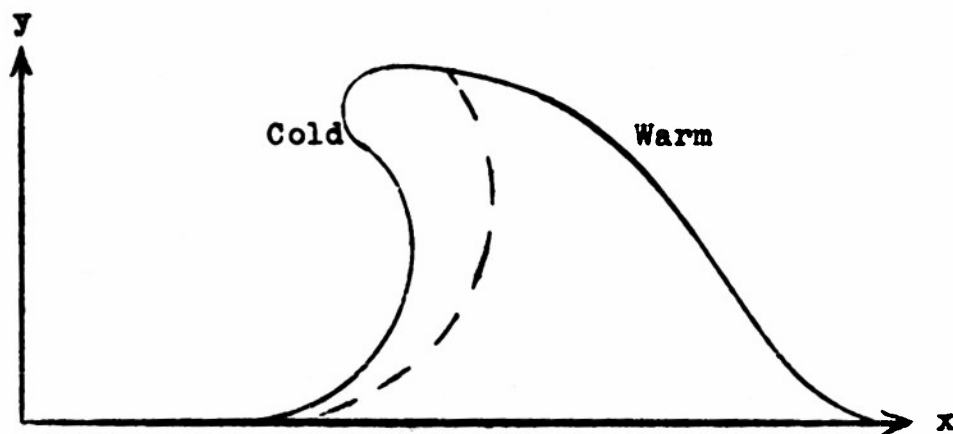$$(20) \qquad\qquad u_f = u_o - 2\sqrt{\alpha k y} ,$$

so that u_f <u>decreases</u> with y. (On the cold front, (20) is gradually modified as explained below, but remains true on the warm front.) Moreover, in the plane y = constant, (13) gives
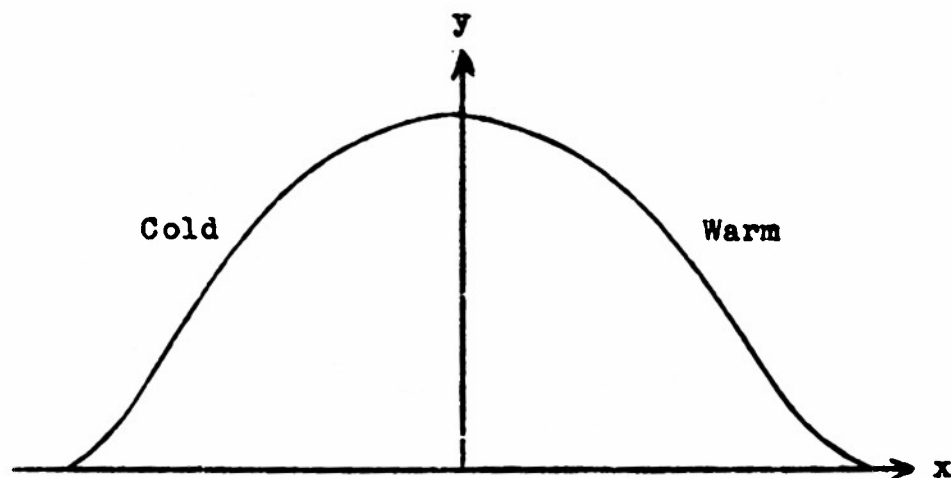
(1)   t = 0
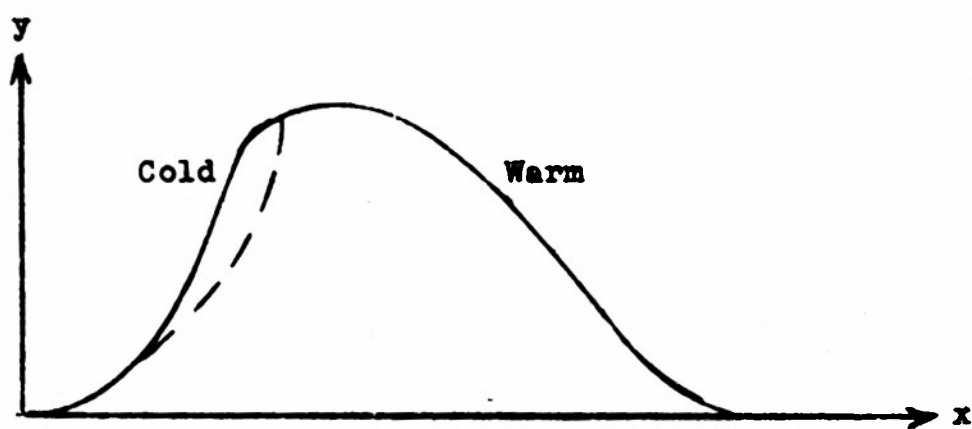
(ii)   t = t₁ > 0

(iii)   t = t₂ > t₁

Fig. 4

$$(21) \qquad x = \xi + \left\{ u_o - 2\sqrt{\alpha\, ky} \right\} t$$

when $C(\xi, y) = 0$; hence, since ξ is the x-coordinate of the front at $t = 0$, (21) shows that the front has moved to the right a distance $\left\{ u_o - 2\sqrt{\alpha\, ky} \right\} t$ at time t. This distance decreases with y; therefore, on the basis of this alone, a warm and cold front system would be distorted for successive times, as shown in Figure 4.

Now, consider how the graph of h against x, in a plane $y = $ constant, varies with t. First take y sufficiently large so that $C(x, y) \neq 0$, for all x; then, initially, the graph is as shown in Figure 5(i). At time t_1, (13) shows
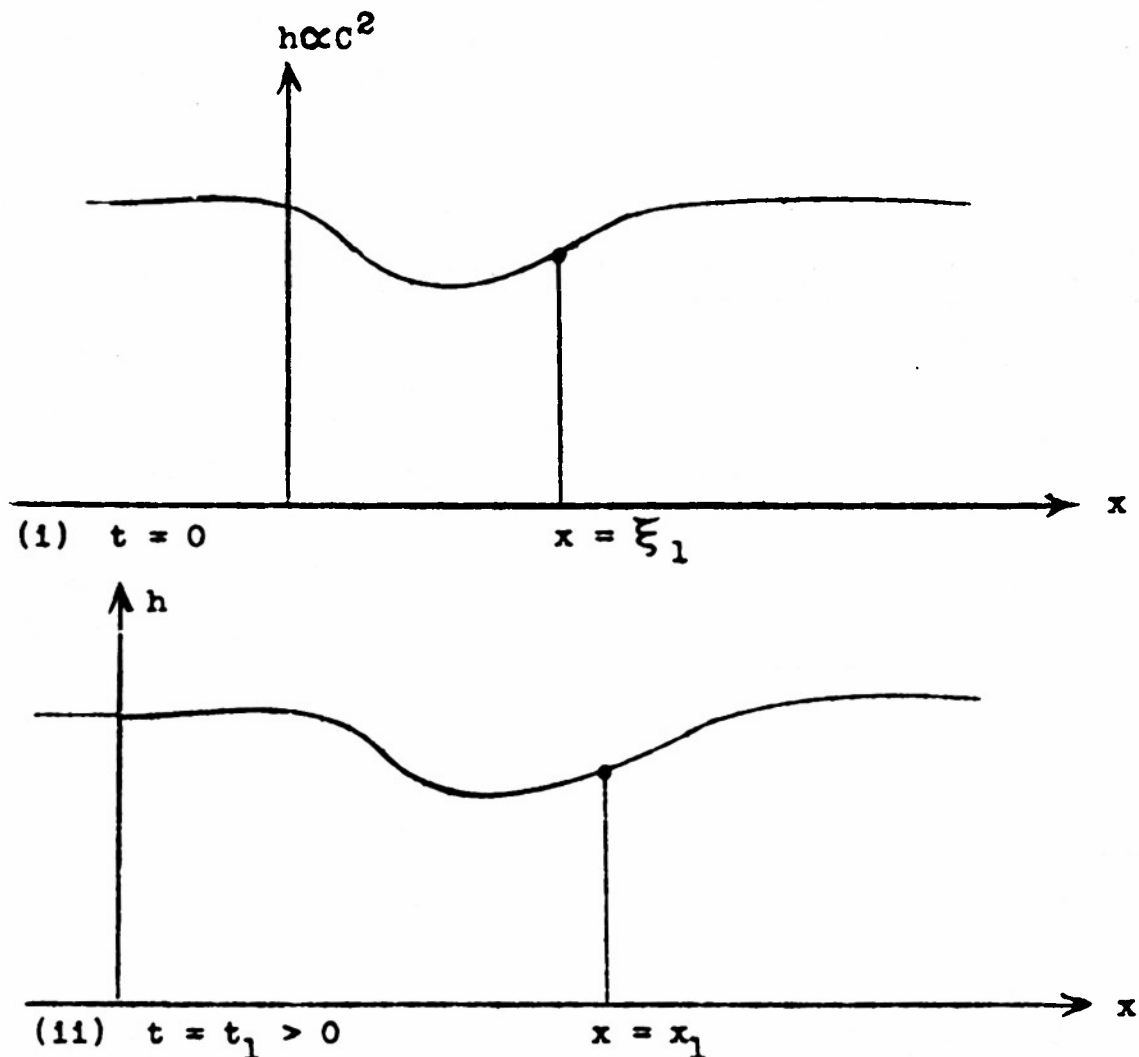

(i) $t = 0$ \qquad $x = \xi_1$

(ii) $t = t_1 > 0$ \qquad $x = x_1$

Fig. 5

that the value of c at the point $x = x_1$ is equal to the value of c which was at the point $x = \xi_1$ at $t = 0$, where $\xi_1 = x_1 - \left\{ A + 3C(\xi_1, y) \right\} t_1$. That is, the value $c = C(\xi_1, y)$ has been displaced to the right by an amount $\left\{ A + 3C(\xi_1, y) \right\} t_1$. Since this quantity is greater for greater values of C, the graph of h becomes distorted as in Figure 5(ii); the "negative region" (where $h_x < 0$) steepens whilst the "positive region" (where $h_x > 0$) flattens out. The positive region continues to smooth out, but, if the steepening of the negative region continued indefinitely, there would ultimately be more than one value of C at the same point, and the wave breaks.[*] Clearly the latter occurs when the tangent at a point of the curve in Figure 5 first becomes vertical. From (15) and (16) we have

$$\frac{\partial c}{\partial x} = \frac{1}{1 + 3C_\xi(\xi, y)t} \, c_\xi(\xi, y);$$

hence, the tangent becomes vertical at $t = (-3C_\xi(\xi, y))^{-1}$. The first breaking occurs on the characteristic $\xi = \xi_m$ for which $|C_\xi(\xi, y)|$ is a maximum, and the time of this first breaking is

$$(22) \qquad\qquad t_m = - \frac{1}{3C_\xi(\xi_m, y)} \,.$$

At this time, the continuous solution breaks down (since c and u would cease to be single-valued functions) and a discontinuous jump in height and velocity must be included. Such a discontinuous "bore" propagates faster than the wavelets ahead of it (the paths of the wavelets in the (x,t)-plane are the characteristics) in a manner analogous to the

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[*]Throughout this work, since the theory bears a close analogy to the theory of waves on shallow water, the terminology of water waves is used in a rather obvious way.

shocks of gas dynamics. The laws of propagation of a bore are obtained by stipulating that the mass of the fluid crossing it is conserved, and the momentum of that fluid is changed only by the difference of the pressure forces acting on either side of the bore; a certain amount of energy is dissipated in turbulence. (The details are given in [6].) The motion of the bore can then be determined by a method which has been used for shocks in problems of gas dynamics (see [8]); only a qualitative description is given here since we are principally interested in events where the frontal surface meets the ground. (However, as suggested by Tepper [7] and discussed by Freeman [3], the discontinuity in the height of the frontal surface may produce a sudden lifting of the air above, resulting in the formation of a squall line.) A bore travels faster than the wavelets ahead and more slowly than the wavelets behind. In the present case, all wavelets in the negative region soon overtake and run into the bore; then the flow behind it is in the uniform undisturbed state. At the same time, the bore slowly penetrates into the positive region (which is continually spreading out); as $t \rightarrow \infty$, the jump in height at the bore tends to zero and in the remaining positive region $h \rightarrow h_o$. The (x, t)-plane is shown in Figure 6 (see next page).

For a plane $y = $ constant in which $C = 0$ does occur, the steepening in the cold front and smoothing out in the warm front take place in a similar way (Figure 7, see next page). Again, the cold front breaks, and, in fact, since C is proportional to the square root of the height, C_ξ is infinite when $C = 0$ so that breaking commences immediately at the ground. But, this time there is an important difference: the breaking cannot be described by a bore since there must be a flux of mass through a bore; hence the cold front moves over the ground, continually breaking and turbulent, in a manner which is difficult to treat theoretically.[*] However

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[*]An approximate treatment of this problem has been developed recently and will be described in a later note.

$$x = (u_0 + c_0)t$$

$$u = u_0, \quad c = c_0$$

$$u = u_0, \quad c = c_0$$

$$x = \xi_0 + (u_0 + c_0)t$$

$$\xi_0$$

Fig. 6

Cold   Warm
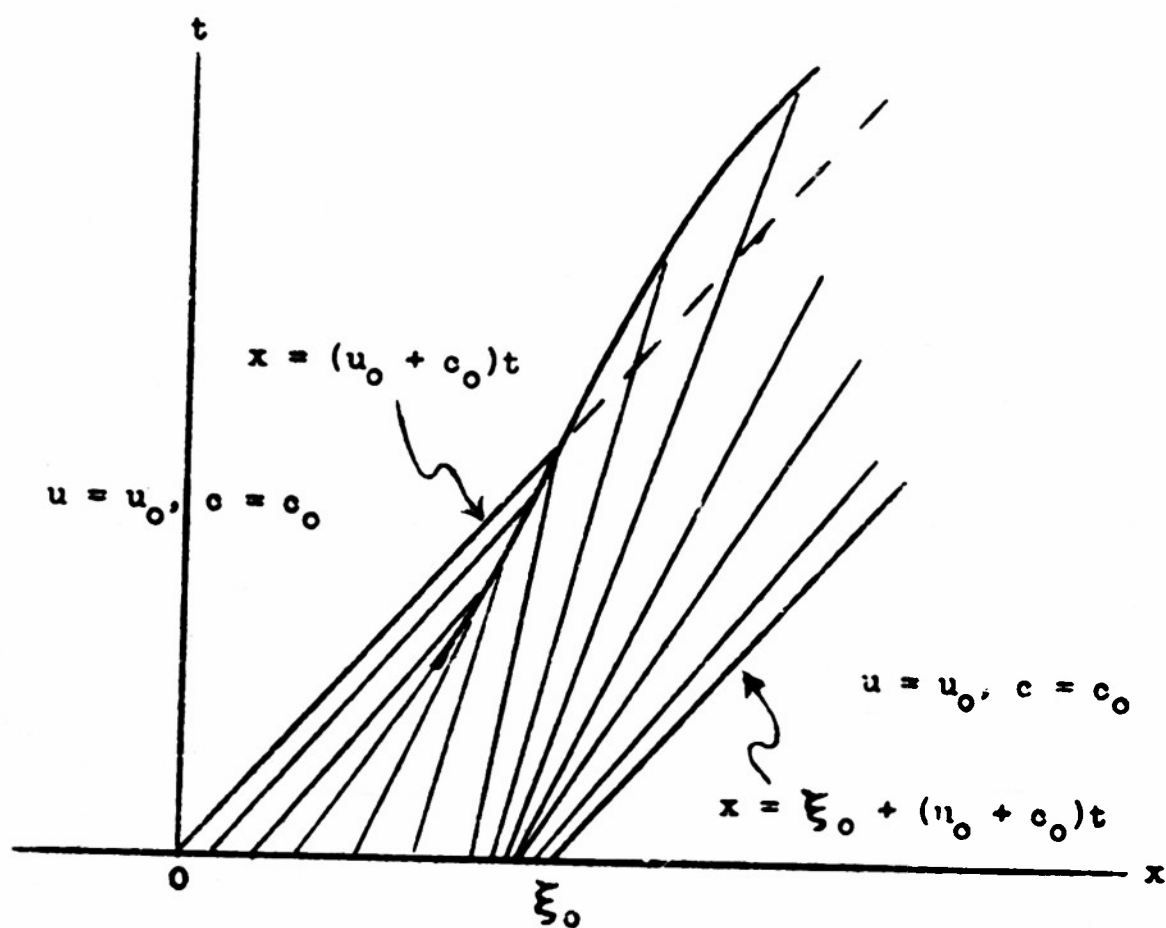
(1) $t = 0$

Cold   Warm
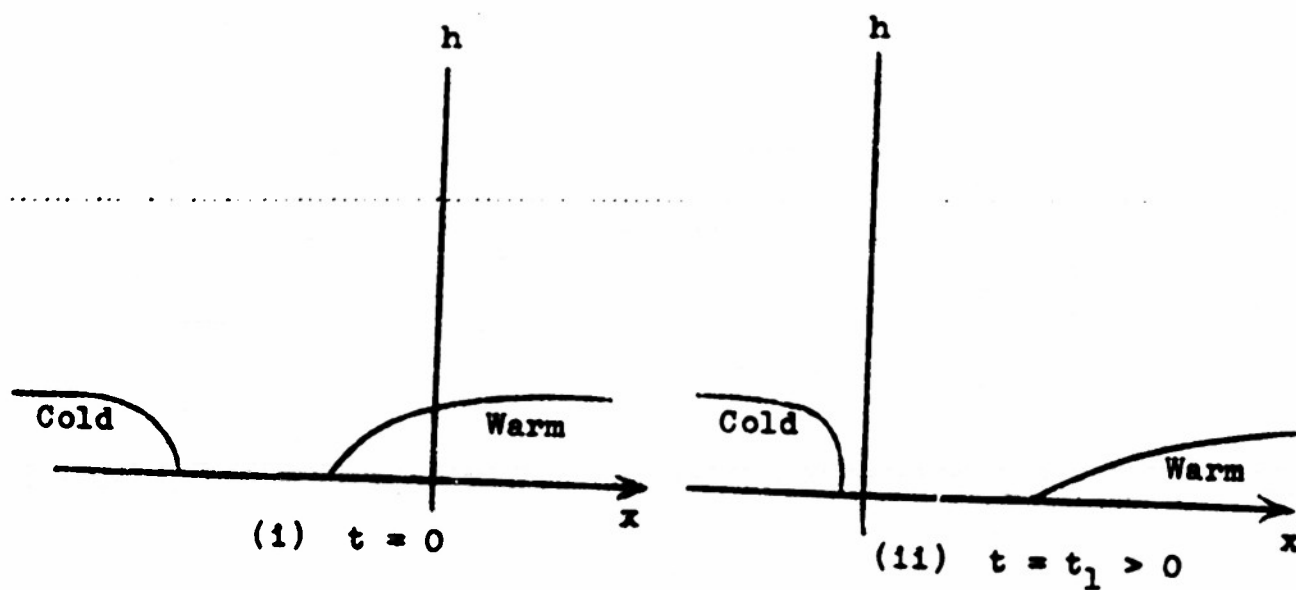
(11) $t = t_1 > 0$

Fig. 7

we can make some deductions about its progress as follows.
Before breaking, the velocities in the cold front region
range from $u_0 - 2c_0$ at the ground to u_0 in the undisturbed
part. In breaking, the later regions carrying the higher
velocities break over the regions of lower velocity; hence,
it is certain that the resulting turbulent front moves
faster than $u_0 - 2c_0$ but slower than u_0. Thus, since the
warm front moves over the ground with speed $u_0 - 2c_0$, the
cold front begins to overtake the warm front and the occlu-
sion process commences.

The breaking of the cold front starts at the ground and
gradually builds up as more of the fluid behind is included.
It would be useful to have even a rough guide to the rate at
which it builds up and overtakes the warm front. After a
time t, breaking will have occurred (in the sense that the
tangent to the surface becomes vertical) at all points for
which $C_\xi(\xi,y) > \frac{1}{3} t$ (compare (22)). Certainly these parts
of the fluid will have been fed into the "breaker," and it
seems reasonable to take the velocity U of the front some-
where between the two extremes $u_0 - 2c_0$ and the velocity on
$\xi = \xi^*$, i.e. $u_0 - 2c_0 + 2C(\xi^*,y)$, where ξ^* is the value
of ξ for which $C_\xi(\xi,y) = \frac{1}{3} t$. A very simple choice would
be to take the mean,

$$(23) \qquad U = u_0 - 2c_0 + C(\xi^*,y),$$

which starts at $u_0 - 2c_0$ (as it should) and increases to-
wards $u_0 - c_0$. (Actually the higher velocities are carried
by deeper sections of fluid, hence the resultant velocity
would be weighted in their favor; on the other hand, energy
is continually being dissipated by turbulence and in over-
coming friction at the ground.) Equation (23) is extremely
rough but it should give the correct order of magnitude for
the variation of U with time, and also something like the

correct dependence upon y. In this way, the intersection of
the frontal surface with the ground would be modified as
shown by the dotted lines in Figure 4 and ultimately the gap
closes up.

Of course, by the time the gap closes, this theory
ceases to describe the motion; nevertheless there is still a
further prediction which can be made. At the center of the
occlusion process (at least initially) the velocity is
$u_0 - 2c_0 \doteq u_0 - 2\sqrt{\alpha k y}$ but near the line of the undisturbed
front $y = 0$, the velocity is u_0. Hence when the gap closes
there would be a tendency to leave a cyclonic (anticlockwise)
rotation around the center of occlusion.

The decrease of the velocity at the fronts with y, the
smoothly propagating warm front, the turbulent cold front,
and the residual cyclonic circulation seem to be in agree-
ment with observation.

Concluding remarks

It is hoped to do further work in this direction which
will include: (1) Application of the approximate theory
(mentioned in the footnote to page 45) of waves breaking
over the ground, to the particular case arising in the
"breaking" cold front of Section 4. This will involve some
numerical integration. (ii) Evaluation of the velocity v
(obtained in Section 3) in some typical case; again numeri-
cal computation is necessary.

If the simple waves, discussed in detail here, are
found to be inadequate for an understanding of the motion,
it will be necessary to solve the approximate equations (5),
(6), and (7) numerically. It may be remarked that the way
in which such a solution would be carried out is quite stan-
dard and well-known, and should not give rise to any addi-
tional difficulties; the difficulty of the wave breaking at
the intersection of the discontinuity surface with the
ground on the cold front side is still expected to arise.

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